A Probabilistic Analysis on Variability of Fatigue Crack Growth Using the Markov Chain

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Understanding the stochastic properties of variability in fatigue crack growth is important to maintaining the reliability and safety of structures. In this study, a stochastic model is proposed to describe crack growth behavior considering the variability of fatigue crack growth rates due to the heterogeneity of material. Fatigue life distribution is then predicted based on this model. To construct this model, fatigue tests are conducted on a high strength aluminum alloy 7075-T6 under constant stress intensity factor range control. The variability of fatigue crack growth rates is expressed by random variables Z and γ based on the variability of material constants C and m of the Paris-Erdogan equation. The distribution of fatigue life under constant stress intensity factor ranges is evaluated by the stochastic Markov chain model based on the Paris-Erdogan equation. The merit of the proposed model is that only a small number of tests are required to determine this function, and fatigue life required to reach certain crack length at a given stress intensity factor range can be easily predicted.

Key Words: Markov Chain Model, Fatigue Crack Growth, Fatigue Life, 2-Parameter Weibull Distribution Function, Stress Intensity Factor Range, Paris-Erdogan equation, Random Variable

Nomenclature

- a : Crack length
- a_o : Initial crack length
- a_f : Final crack length
- b : State number
- B : Thickness
- C, m : Material constants in Paris-Erdogan equation
- C_o, m_o : Expected values of C and m
- ΔK : Stress intensity factor range
- N : Number of cycles
- $\boldsymbol{P}$: Transition matrix
- p : Transition probability
- $\boldsymbol{P}_o$: Initial probability vector
- $\boldsymbol{P}_x$: Probability vector
- s : Number of specimen
- U : Random number
- Z, γ : Random variables according to material constants C and m
- α, β : Parameters of the 2 parameter Weibull distribution

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1. Introduction

Understanding the sources and mechanisms of damage due to fatigue loading in engineering design, is important to mainatin reliability and safety in machinery structures. Much experimental data is therefore necessary in order to evaluate the characteristics of the fatigue process and ensure safety during service loading.

Typically, experimental investigations of fatigue crack growth under constant amplitude cyclic loading have been performed to find the curves relating fatigue crack length a to the number of cycles N. However, the crack growth process contains various physical uncertainties caused by the heterogeneity of materials. Therefore, many attempts have been recently made to formulate stochastic models for fatigue crack growth phenomena and to clarify the properties associated with them.

These stochastic models may be separated into two types: models which are derived from a randomization of the Paris-Erdogan equation (1963), which is well known as a fundamental law for fatigue crack growth (Lin and Yang, 1985; Ishikawa and Tsurui, 1987; Yoon, Yang and Yoon, 1992), and models that analyze the stochastic nature of fatigue crack growth using the Markov chain or the Markov process (Bogdanoff, 1978; Bogdanoff and Kozin, 1980; Yoshio, Takao and Hisanobu, 1983; Kozin and Bogdanoff, 1983; Lise, Rune and Lara, 1991; Kim and Kim, 1995)

In the first type, a generalized Fokker-Planck equation is derived by randomizing the empirical Paris-Erdogan equation, which describes the temporal variability of crack length distribution (Lin and Yang, 1985; Ishikawa and Tsurui, 1987). Using the solution of this equation, crack growth life distribution can be determined. This method seems to be a very reasonable one to analyze crack growth including physical uncertainties. However, a sample process for crack growth is not explicitly derived.

The later type is discussed by Bogdanoff and Kozin. (1978, 1980, 1983). In these models, crack growth is described by the discrete Markov chain (Lise, Rune and Lara, 1991; Kim and Kim, 1995), and the life distribution and the sample process for crack growth are obtained by using the transition probability matrix of the Markov chain. However, since this approach is more closely related to statistical analysis than to the particular problem in fracture mechanics, the physical meanings of the model seem unclear.

To evaluate the variability of fatigue crack growth, much data must be acquired experimentally. The purpose of this paper is to present a stochastic model which requires only a small number of tests to describe the variability of fatigue crack growth. Fatigue tests are carried out under constant stress intensity factor range control, and the variability of fatigue crack growth rates is investigated by estimating the variability of material constants C and m in the Paris-Erdogan equation from experiments.

2. Theoretical Backgroud

Generally, fatigue crack growth is well described by the Paris-Erdogan equation, which is based upon the principle of fracture mechanics:

$$\frac{da}{dN}=C(\Delta K)^m \tag{1}$$

where C and m are material constants that are determined experimentally.

However, material constants C and m should be treated as variables along the crack path because of dispersion in experimental results:

$$\frac{da}{dN}=C(x)(\Delta K)^{m(x)} \tag{2}$$

In the Markov chain model, the damage is assumed to progress along steps of the crack length δa. Thus the ith state of damage can be defined as

$$a_i=a_o+i\delta a \quad i=0,\ 1,\ 2,\ 3,\ \cdots,\ b \tag{3}$$

and crack growth rates can be given by

$$\frac{\delta a}{\delta N_i}=C(x)(\Delta K_i)^{m(x)} \tag{4}$$

Introducing random variables Z and r based on the variability of material constants C and m in the Paris-Erdogan equation, the number of cycles, δN_i, that is used to propagate the crack one step, δa, is given as

$$\frac{\delta a}{\delta N_i}=ZC_o(\Delta K_i)^{rm_o} \tag{5}$$

or

$$\delta N_i=\frac{\delta a}{ZC_o(\Delta K_i)^{rm_o}} \tag{6}$$

The expected value of δN_i is

$$E[\delta N_i]=\frac{\delta a}{C_o(\Delta K_i)^{m_o}} \tag{7}$$

Let us assume that the first $(\delta n-1)$ duty cycles do not lead to crack growth, with probability $p^{(\delta n-1)}$ whereas the δnth duty cycle results in crack growth, with probability q. Thus, the probability distribution of δN is a geometric distribution (Bogdanoff, 1978) given as

$$P[\delta N=\delta n]=qp^{\delta n-1} \tag{8}$$

The expected value and variance of the number of duty cycles are given by the first and the second moments of the geometric distribution (Bogdanoff. 1978), respectively:

$$E[\delta N_i]=\frac{\delta a}{C_o(\Delta K_i)^{m_o}}=\frac{1}{q_i} \tag{9}$$

$$Var[\delta N_i]=E[\delta N_i^2]-(E[\delta N_i])^2=\frac{1-q_i}{q_i^2} \tag{10}$$

The state number b can also be obtained from $E[\delta N_i]$ and $Var[\delta N_i]$:

$$b=\frac{(1-q)(E[\delta N_i])^2}{Var[\delta N_i]} \tag{11}$$

From the discrete Markov chain model, the transition matrix $\boldsymbol{P}$ is obtained from the transition probability $q_i(q_i=1-p_i)$ for propagating the crack one step δa:

$$\boldsymbol{P}=\begin{bmatrix} p_1 & q_1 & 0 & \cdots & 0 \\ 0 & p_2 & q_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{12}$$

where $p_i>0$, $p_i+q_i=1$

Let the initial state of damage be specified by the $(1\times b)$ row vector $\boldsymbol{P}_o$.

$$\boldsymbol{P}_o=[p_o(1),\ p(2),\ p_o(3),\ \cdots,\ p_o(b)] \tag{13}$$

$$\sum_{i=1}^{b} p_o(i)=1$$

where, $p_o(i)$ is the probability of being in the ith state of damage.

From Markov chain theory, the probability vector $\boldsymbol{P}_x$ of being in the xth step is written as

$$\boldsymbol{P}_x=\boldsymbol{P}_o\boldsymbol{P}^x \tag{14}$$

3. Experimental Procedure

3.1 Material and Specimen

The material used in this experiment is a high strength aluminum alloy 7075-T6. The chemical composition and the mechanical properties of this material are tabulated in Table 1 and Table 2, respectively. According to ASTM E647-93 (1993), Compact tension (CT) specimens of 50.8 mm ligament ($W=50.8$ mm) are used in this study. To investigate the effect of thickness on variability, specimens having thickness B=3.2 and 25.4 mm were used. The tensile axis was parallel to the rolling direction (L-T).

Table 1 Chemical compostion (wt%).

Designation	Si	Fe	Cu	Mn	Mg	Cr	Zn	Al
7075-T6	0.10	0.38	1.25	0.14	9.15	0.22	7.30	Re.

Table 2 Mechanical properties of 7075-T6 Al-alloy.

Yield strength (MPa)	Tensile strength (MPa)	Elongation (MPa)
461.9	524	12.9

Crosshead speed : 1 mm/min
Specimen thickness : 3.2 mm

3.2 Fatigue test

According to ASTM E 647-93(1993), stress intensity factor range control fatigue tests were conducted in air at room temperature on a servo-hydraulic material testing machine (MTS, model 458. 91) having a load capacity of 10 tons. Sinusoidal cyclic loading of stress ratio ($R=\sigma_{min}/\sigma_{max}$) 0.3 and loading frequency 10 Hz was applied at three different stress intensity factor range levels. The stress intensity factor ranges selected were $\Delta K=6.5\text{MPa}\sqrt{\text{m}}$, $8.4\text{MPa}\sqrt{\text{m}}$ and $10.3\text{MPa}\sqrt{\text{m}}$. During cyclic loading the crack length was monitored by the compliance method, and test data were automatically acquired with a measurement system controlled by PC as the crack length increase 0. 1 mm.

4. Experimental Results and Discussion

4.1 The variability in fatigue crack growth

Figure 1 shows the relationship between crack length a and the number of cycles N under constant stress intensity factor range ΔK. Though ΔK has a constant value, a difference in inclination exists in each curve. It means that crack growth rates da/dN differ from specimen to specimen due to initial damage state, though they are of the same material. Figure 2 presents the

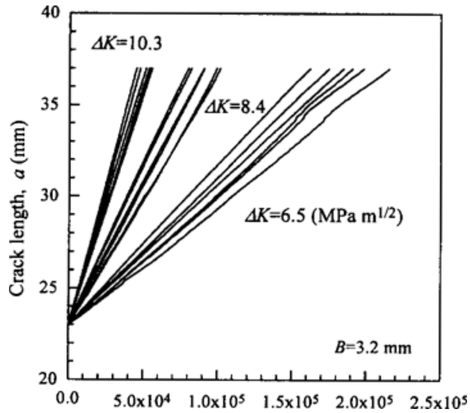


Fig. 1 $a-N$ curves under constant stress intensity factor range.

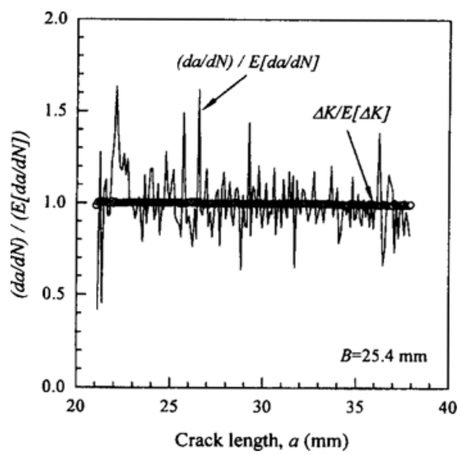


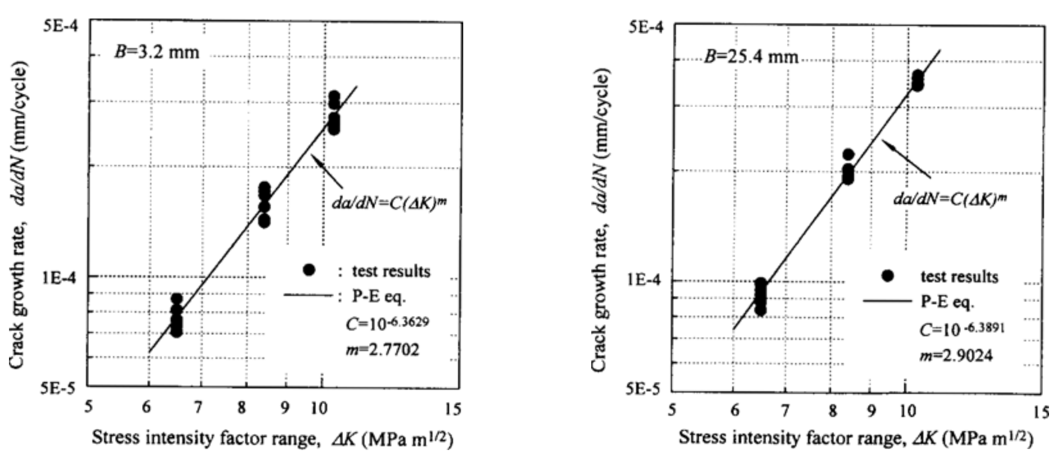
Fig. 2 Example of crack growth rate along the crack path.

states of stress intensity factor range control and the variability of crack growth rates along the crack growth path. The range of stress intensity factor is well controlled, but the crack growth rates show remarkable fluctuations due to the heterogeneity of materials.

Figure 3 (a) and (b) show the relationship between da/dN and ΔK in the case of $B=3.2$ mm and $B=25.4$ mm, respectively. Crack growth rates grow linearly as the stress intensity factor range increases. This relationship is well described by the Paris-Erdogan equation. However, crack growth rates fluctuate even under constant ΔK. Thus, a statistical model is needed to describe the scattering of crack growth rates.

4.2 The stochastic properties of fatigue crack growth

Suppose the variability of crack growth rates due to the heterogeneity of materials depends on the material constants C and m of the Paris-Erdogan equation. These values should be treated as variables along the crack path. Therefore the material constants C and m can be written as $C(x)$ and $m(x)$ at crack length x, respectively. Figure 4 (a), (b) show $C(x)$ and $m(x)$ along the crack path in the case of thickness 3.2 mm and 25.4 mm, which are estimated from the test results of the crack growth rates versus stress intensity factor range at crack length x using least-square linear regression. The band of variability of



(a) Thickness 3.2 mm (b) Thickness 25.4 mm

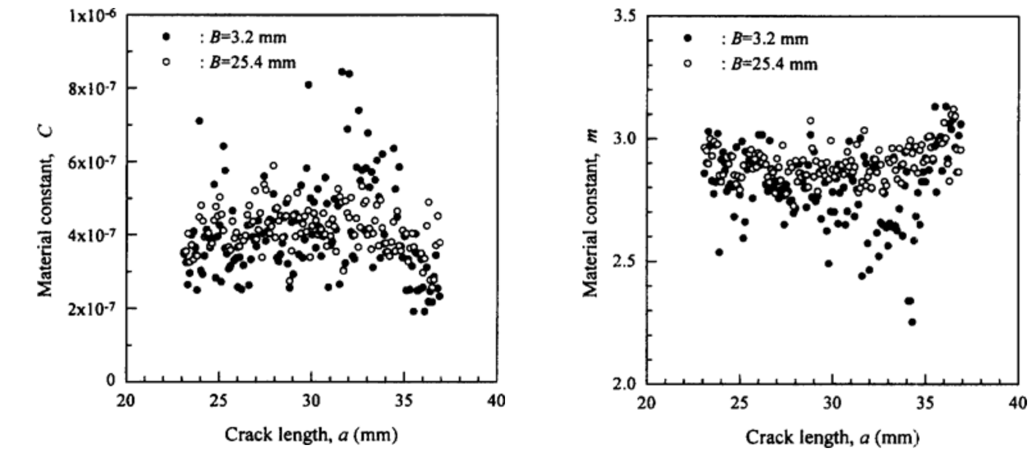
Fig. 3 Relationship between crack growth rates and stress intensity factor range.

material constants $C(x)$ and $m(x)$ for in thickness 3.2 mm is larger than the band of variability for thickness 25.4 mm.

$C(x)$ and $m(x)$ fluctuate randomly because the crack growth rates vary in spite of a constant stress intensity factor range. Therefore, random variables Z and γ are introduced to consider the variability of fatigue crack growth rates.

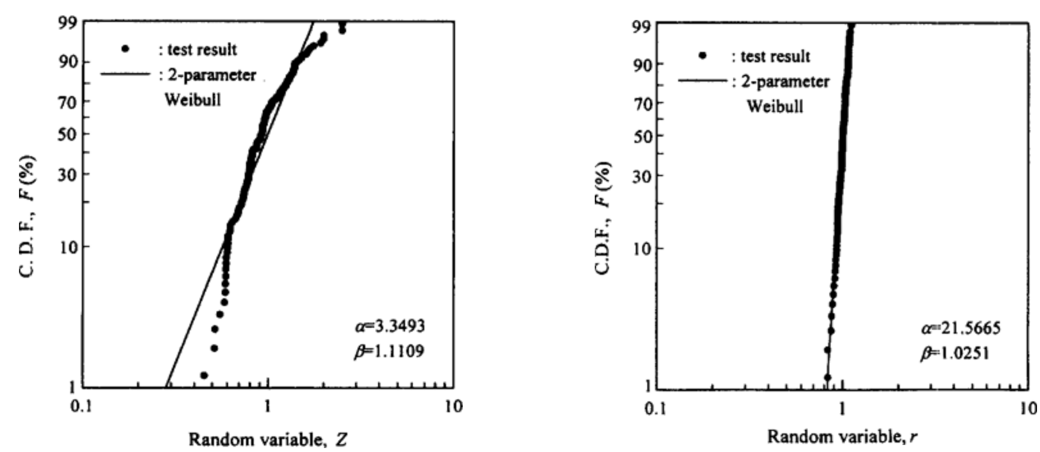
$$Z=\frac{C(x)}{C_o} \text{ where } C_o=E[C(x)] \tag{15}$$

$$r=\frac{m(x)}{m_o} \text{ where } r_o=E[r(x)] \tag{16}$$



(a) Distribution of material constant C (b) Distribution of material constant m

Fig. 4 Distribution of material constants.



(a) Random variable, Z for the variability of C (b) Random variable, r for the variability of m

Fig. 5 Distribution of random variables using the 2-parameter Weibull (B=3.2 mm).

Table 3 The value of each parameter in 2-parameter Weibull function.

B (mm)	C_o	m_o	Z			r		
			α	β	$Var[Z]$	α	β	$Var[r]$
3.2	4.3361E-7	2.7702	3.3492	1.1109	0.1702	21.5665	1.0251	0.0032
25.4	4.0823E-7	2.9024	8.3710	1.0592	0.0209	49.0847	1.0114	0.0006

Z and γ are the variables according to the fluctuation of material constant C and m, respectively.

Figure 5 (a) and (b) present the characteristic of the variables Z and γ evaluated by the 2-parameter Weibull distribution function. The values of the parameters are tabulated in Table 3.

$$F(t)=1-\exp\left(-\left(\frac{t}{\beta}\right)^{\alpha}\right) \qquad (17)$$

where, β is a scale parameter and α is a shape parameter.

As shown in these figures, the variables Z and γ are well described by this distribution function. Therefore, the variables Z and γ can be defined as the random variables according to the 2-parameter Weibull distribution.

Figure 6 shows the relationship between the random variables Z and γ. They are correlated with each other linearly, and the band of variability increases as the thickness decreases.

4.3 The prediction of fatigue life

Considering these properties of variability in fatigue crack growth, the prediction of fatigue life under the condition characterizing the experiments is performed by a discrete Markov chain model. The flowchart is given Fig. 7. Assuming the increase of crack length 0.1 mm as one step, and the duty cycle as 100 cycles at the given stress intensity factor range, a simulation is carried out.

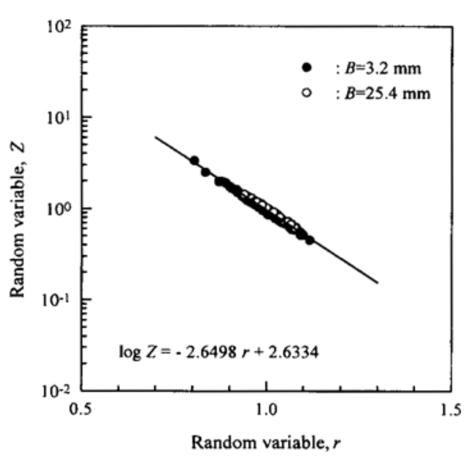


Fig. 6 Relationship between random variables Z and γ.

The initial probability vector is given as follows:

$$\boldsymbol{P}_o=[1,\ 0\ 0,\ \cdots\cdots,\ 0] \qquad (18)$$

Figure 8 shows a-N curves simulated by this

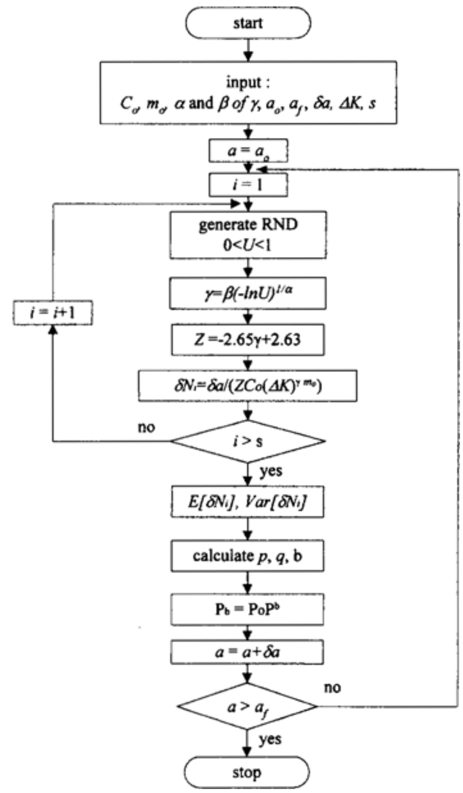


Fig. 7 Flowchart of simulation by this stochastic analysis.

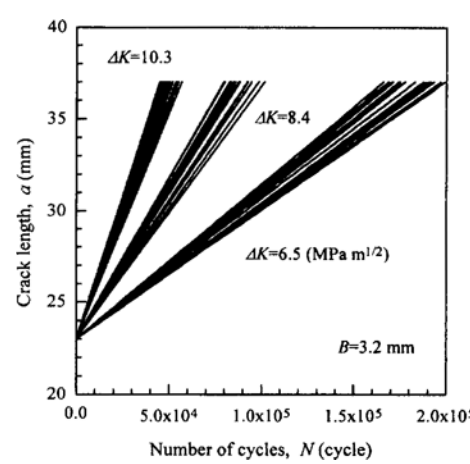
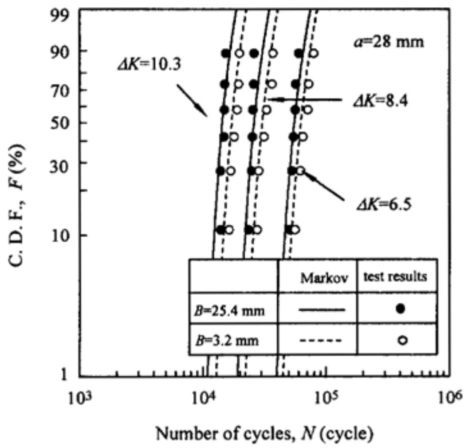
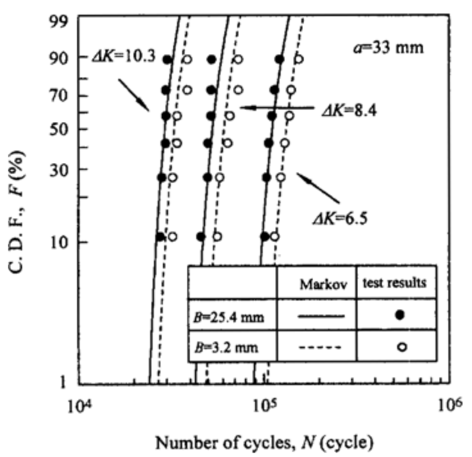


Fig. 8 $a-N$ curve simulated by the Markov chain model (50 samples).



(a) Crack length $a=28$ mm



(b) Crack length $a=33$ mm

Fig. 9 Comparison between the test results and the predicted fatigue lives by the proposed model.

model in the case of thickness 3.2 mm. Using the algorithm for generating random numbers from the 2-parameter Weibull distribution, random variables Z and r were obtained. As shown in this figure, the results describe well the experimental results in Fig. 1. Figure 9 (a) and (b) present the probability distributions of the number of cycles to reach crack lengths 28 and 33 mm. Solid and dashed curves are the probability distribution predicted by this model. The corresponding experimental results coincide well with these curves.

5. Conclusions

In this paper, we investigated the stochastic properties in the variability of fatigue crack growth rates on 7075-T6 Al-alloy and developed a stochastic model. We then predicted fatigue lives which reached certain crack length under the given stress intensity factor ranges with this model. The variability of fatigue crack growth rates can be expressed by random variables Z and r based on the variability of material constants C and m in the Paris-Erdogan equation. Random variables Z and r were well described by the 2-parameter Weibull distribution function and correlated with each other linearly. From these relationships, a stochastic model was developed from the Markov chain model.

The distributions of fatigue lives with respect to the stress intensity factor range were evaluated by the stochastic Markov chain model based on the Paris-Erdogan equation. The results showed good agreement with experimental results. The merits of the proposed model are that only a small number of tests are required to determine this function, and fatigue crack growth life required to reach certain crack length is easily predicted at the given stress intensity factor range.

Acknowledgements

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